MATHEMATICS

AN EXTENSION OF A THEOREM OF P. P. KOROVKIN TO SINGULAR INTEGRALS WITH NOT NECESSARILY POSITIVE KERNELS

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ABSTRACT

Concerning the approximation of periodic functions by means of singular convolution integrals having positive trigonometric kernels, an equivalence theorem of P. P. Korovkin on convergence factors, i.e., on the Fourier coefficients of the kernel, is of great importance. A simple inversion formula between convergence factors and trigonometric moments of a general kernel immediately delivers an extension of this theorem to a class of singular integrals with not necessarily positive kernels as well as a simplified proof of the original result. The new theorem is applied to three representative examples of oscillating kernels.

In this note *) we consider a special feature of the approximation of periodic functions f belonging to $C_{2\pi}$ (or $L^p_{2\pi}$, $1 \leqslant p < \infty$) by means of singular integrals of convolution type

$$(1) \qquad I_\xi(\mathfrak{p}; f; x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x-t) p_\xi(t)\, dt \qquad (\xi \to \xi_0).$$

It is assumed that the underlying even kernel has the Fourier series representation

$$(2) \qquad p_\xi(x) = \tfrac{1}{2} + \sum_{k=1}^{\infty} \varrho_{k,\xi}(\mathfrak{p}) \cos kx$$

normalized by $\int_{-\pi}^{\pi} p_\xi(t) dt = \pi$ for all ξ. The convergence factors are given by

$$(3) \qquad \varrho_{k,\xi}(\mathfrak{p}) = \frac{2}{\pi} \int_0^{\pi} p_\xi(t) \cos kt\, dt \qquad (k = 0, 1, 2, \ldots)$$

i.e., the real Fourier coefficients of (2). (In case the parameter ξ is discrete, say $n = 1, 2, 3, \ldots$ with $n \to \infty$, the kernel becomes a trigonometric polynomial of degree n with $\varrho_{k,n}(\mathfrak{p}) = 0$ for $k > n$.) Furthermore it is assumed that the linear operator (1) is a strong approximation process, i.e.,

$$\lim_{\xi \to \xi_0} \| I_\xi(\mathfrak{p}; f; x) - f(x) \| = 0$$

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*) Extracted from the first part of the author's dissertation [14] prepared under the direction of Prof. P. L. Butzer. The author is indebted to Prof. Butzer and Dr. W. Trebels for many fruitful discussions and their permanent attention.

for all $f \in C_{2\pi}$ (or $L^p_{2\pi}$). Necessary and sufficient conditions for this to be the case are (with $L_\xi(\mathfrak{p})$ being the associated Lebesgue constants and M a positive constant)

$$L_\xi(\mathfrak{p}) = \int_0^\pi |p_\xi(t)|dt \leqslant M \qquad (\xi \to \xi_0),$$

$$\lim_{\xi \to \xi_0} \varrho_{k,\xi}(\mathfrak{p}) = 1 \qquad (k = 1, 2, 3, \ldots) \tag{4}$$

(see e.g. [1]). Hence all kernels below are, a priori, assumed to generate strong approximation processes.

Under these hypotheses there is a most interesting equivalence theorem of P. P. Korovkin [7] on the convergence factors of positive kernels.

Theorem (P. P. Korovkin). *If $p_\xi(x) \geqslant 0$, then for all $k = 2, 3, 4, \ldots$*

$$\lim_{\xi \to \xi_0} \frac{1-\varrho_{k,\xi}(\mathfrak{p})}{1-\varrho_{1,\xi}(\mathfrak{p})} = k^2 \Leftrightarrow \lim_{\xi \to \xi_0} \frac{1-\varrho_{2,\xi}(\mathfrak{p})}{1-\varrho_{1,\xi}(\mathfrak{p})} = 4. \tag{5}$$

In other words, the limit on the left side which plays an important rôle in approximation theory is valid for *all* $k = 2, 3, 4, \ldots$ provided it only holds for $k = 2$.

It is the purpose of this note to generalize this result to the case that the kernels are not necessarily positive as well as to give a new elementary proof of the original theorem of P. P. Korovkin.

Denoting the trigonometric moment of order 2σ of $p_\xi(x)$ by

$$\begin{cases} T(\mathfrak{p}; 2\sigma; \xi) = \dfrac{2}{\pi} \displaystyle\int_0^\pi \left(2 \sin \frac{t}{2}\right)^{2\sigma} p_\xi(t)dt \\ \qquad = I_\xi\left(\mathfrak{p}; \left(2 \sin \dfrac{\cdot}{2}\right)^{2\sigma}; 0\right) \qquad (\sigma = 0, 1, 2, \ldots), \end{cases} \tag{6}$$

our first result is concerned with an explicit representation of these moments and a corresponding inversion formula.

Lemma. *The following relations hold:*

$$T(\mathfrak{p}; 2\sigma; \xi) = 2 \sum_{k=1}^{\sigma} (-1)^{k+1} \binom{2\sigma}{\sigma - k} (1 - \varrho_{k,\xi}(\mathfrak{p})) \qquad (\sigma = 1, 2, 3, \ldots), \tag{7}$$

$$1 - \varrho_{k,\xi}(\mathfrak{p}) = \sum_{\sigma=1}^{k} (-1)^{k+1} \frac{k}{k+\sigma} \binom{k+\sigma}{2\sigma} T(\mathfrak{p}; 2\sigma; \xi) \qquad (k = 1, 2, 3, \ldots). \tag{8}$$

Proof. From formula [11, p. 26]

$$\sin^{2n} x = 2^{-2n} \left\{ 2 \sum_{k=0}^{n-1} (-1)^{n-k} \binom{2n}{k} \cos 2(n-k)x + \binom{2n}{n} \right\}$$

it follows that

$$(9)\qquad \left(2\sin\frac{t}{2}\right)^{2\sigma}=2\left\{\frac{1}{2}\binom{2\sigma}{\sigma}-\sum_{j=1}^{\sigma}(-1)^{j+1}\binom{2\sigma}{\sigma-j}\cos jt\right\}.$$

This immediately yields, using (3) and (6), that

$$(10)\qquad T(\mathfrak{p};2\sigma;\xi)=2\left\{\frac{1}{2}\binom{2\sigma}{\sigma}-\sum_{j=1}^{\sigma}(-1)^{j+1}\binom{2\sigma}{\sigma-j}\varrho_{j,\xi}(\mathfrak{p})\right\},$$

and setting $t=0$ in (9)—see (23)—this delivers (6).

The inversion formula (8) is a direct consequence of definition (3) and

$$(11)\qquad 1-\cos kt=\sum_{\sigma=1}^{k}(-1)^{\sigma}\frac{k}{k+\sigma}\binom{k+\sigma}{2\sigma}\left(2\sin\frac{t}{2}\right)^{2\sigma}\qquad (k=1,2,3,\ldots).$$

Since this representation of $\cos kt$ as a linear combination of powers of $(2\sin(t/2))^2$ is fundamental in this connection, one possible derivation of (11) may be indicated as follows. From [13, p. 124 ff, (48), (a), (b)]

$$\cos m\vartheta=\sum_{n=0}^{m/2}\frac{(-1)^n}{(2n)!}\prod_{k=0}^{n-1}[m^2-(2k)^2]\sin^{2n}\vartheta \qquad (m\ \textit{even}),$$

$$\prod_{k=0}^{n-1}[m^2-(2k)^2]_{|n=0}=1$$

it follows that $(m=2k;\ \vartheta=t/2)$

$$(12)\qquad 1-\cos kt=\sum_{\sigma=1}^{k}(-1)^{\sigma+1}\left\{\frac{1}{(2\sigma)!}\prod_{j=0}^{\sigma-1}(k^2-j^2)\right\}\left(2\sin\frac{t}{2}\right)^{2\sigma}.$$

By means of the product formula

$$(13)\qquad \frac{1}{(2\sigma)!}\prod_{j=0}^{\sigma-1}(k^2-j^2)=\frac{k}{k+\sigma}\binom{k+\sigma}{2\sigma}\qquad (\sigma,k=1,2,3,\ldots)$$

which is easily verified, (12) finally passes into (8).—Since $T(\mathfrak{p};0;\xi)\equiv 1$ the convergence factors in (8) may be written as

$$\varrho_{k,\xi}(\mathfrak{p})=\sum_{\sigma=0}^{k}(-1)^{\sigma}\frac{k}{k+\sigma}\binom{k+\sigma}{2\sigma}T(\mathfrak{p};2\sigma;\xi)\quad (k=0,1,2,\ldots);$$

this may in turn be compared with the original definition (3).

Formula (10) has been applied in a similar fashion for the first time by È. A. Komleva [5, p. 90 (1.4); 6] (cf. also [8, p. 12; 9, p. 1437 f], [10, p. 26 f], [12, p. 189 (4)]) in order to establish higher order Voronovskaja type expansions for positive operators (1), i.e., with positive kernels (2); concerning these problems compare e.g. [2] and the literature cited there. Formula (8) seems to be new.

Whereas the relations of the Lemma reveal a certain symmetry using the binomial coefficients, the importance of (8) will be seen, however, by going back to (13) from

COROLLARY 1. *For arbitrary kernels $p_\xi(x)$ one has*

$$(14)\quad 1-\varrho_{k,\xi}(\mathfrak{p})=\sum_{\sigma=1}^{k}(-1)^{\sigma+1}\{\prod_{j=0}^{\sigma-1}(k^2-j^2)\}\frac{1}{(2\sigma)!}T(\mathfrak{p};2\sigma;\xi)\quad (k=1,2,3,\ldots).$$

The fact that the product in (14) is an even algebraic polynomial of the integer variable k of degree 2σ without constant term, namely

$$A_\sigma(k)\equiv\prod_{j=0}^{\sigma-1}(k^2-j^2)=\sum_{j=1}^{\sigma}\alpha_{j\sigma}k^{2j}\qquad(\alpha_{j\sigma}\text{ real};\ \alpha_{\sigma\sigma}=1;\ \sigma=1,2,3,\ldots)$$

with

$$A_1(k)=k^2,\ A_2(k)=k^4-k^2,\ A_3(k)=k^6-5k^4+4k^2,\ \ldots$$

has deeper consequences.

For this fact, under the assumption $T(\mathfrak{p};2;\xi)=2(1-\varrho_{1,\xi}(\mathfrak{p}))\neq 0$, the following identity is an obvious example

$$(15)\qquad \frac{1-\varrho_{k,\xi}(\mathfrak{p})}{1-\varrho_{1,\xi}(\mathfrak{p})}=k^2-2\sum_{\sigma=2}^{k}(-1)^\sigma A_\sigma(k)\frac{1}{(2\sigma)!}\frac{T(\mathfrak{p};2\sigma;\xi)}{T(\mathfrak{p};2;\xi)};$$

on the right-hand side only the quotient of the moments depends upon ξ. But now (15) delivers the main

THEOREM. *For arbitrary kernels $p_\xi(x)$ with first convergence factor $\varrho_{1,\xi}(\mathfrak{p})\neq 1$ the following two assertions are equivalent:*

$$(16)\qquad \lim_{\xi\to\xi_0}\frac{1-\varrho_{k,\xi}(\mathfrak{p})}{1-\varrho_{1,\xi}(\mathfrak{p})}=k^2\qquad (k=2,3,4,\ldots),$$

$$(17)\qquad \lim_{\xi\to\xi_0}\frac{T(\mathfrak{p};2\sigma;\xi)}{T(\mathfrak{p};2;\xi)}=0\qquad (\sigma=2,3,4,\ldots).$$

Here it is by no means necessary that the kernel is positive.

In the particular case of positive kernels, thus with Lebesgue constant $L_\xi(\mathfrak{p})\equiv\pi/2$ for all ξ and (4) replaced by merely $\lim_{\xi\to\xi_0}\varrho_{1,\xi}(\mathfrak{p})=1$, the Theorem reduces to Korovkin's theorem stated above. An enlarged version, including (5), reads as

COROLLARY 2. *For kernels $p_\xi(t)\geqslant 0$ the following assertions are equivalent:*

$$(18)\qquad \lim_{\xi\to\xi_0}\frac{1-\varrho_{k,\xi}(\mathfrak{p})}{1-\varrho_{1,\xi}(\mathfrak{p})}=k^2\qquad (k=2,3,4,\ldots),$$

$$(19)\qquad \lim_{\xi\to\xi_0}\frac{1-\varrho_{2,\xi}(\mathfrak{p})}{1-\varrho_{1,\xi}(\mathfrak{p})}=4,$$

$$(20)\qquad \lim_{\xi\to\xi_0}\frac{T(\mathfrak{p};4;\xi)}{T(\mathfrak{p};2;\xi)}=0,$$

$$(21)\qquad \lim_{\xi\to\xi_0}\frac{T(\mathfrak{p};2\sigma;\xi)}{T(\mathfrak{p};2;\xi)}=0\qquad (\sigma=2,3,4,\ldots).$$

PROOF. The equivalence of (19) and (20) is nothing but (15) in case $k=2$, thus

$$\frac{1-\varrho_{2,\xi}(\mathfrak{p})}{1-\varrho_{1,\xi}(\mathfrak{p})}=4-\frac{T(\mathfrak{p};4;\xi)}{T(\mathfrak{p};2;\xi)}.$$

Conclusion (20) $\Rightarrow$ (21) follows from the estimate

$$T(\mathfrak{p};2(\sigma+\sigma');\xi)<2^{2\sigma'}T(\mathfrak{p};2\sigma;\xi) \qquad (\sigma,\sigma'=1,2,3,\ldots). \tag{22}$$

The latter is obvious for positive kernels from definition (6) since then, in addition to (20) in case $\sigma=2$, even

$$T(\mathfrak{p};2\sigma';\xi)=o(T(\mathfrak{p};2;\xi)) \qquad (\sigma'=3,4,5,\ldots)$$

holds for the higher order moments, too. The final step (21) $\Rightarrow$ (18) is then immediately established again by (15).

Thus we proved Corollary 2, in contrast to [7], without using any information concerning the approximation behaviour of the corresponding singular integral (1) as, for instance, a Voronovskaja type theorem. It follows merely from the trivial but decisive estimate (22). In comparison with the Theorem it is seen that the equivalences (19), (20) turn out to be very specific for positive kernels and are lost under the more general hypotheses of the Theorem.—Another quite different direct proof for the equivalence (5) is given in [4, p. 216 f]. For the importance of Corollary 2 in several branches of approximation theory concerning positive operators we again refer to [2] (cf. [3], [1]), where also a detailed list of sources is to be found.

Concerning the Theorem, we shall finally discuss three representative examples of oscillating kernels, namely three with at least two (symmetrical) changes of sign (zeros of odd multiplicity) in $(-\pi,\pi)$. The verification of (17) will bring forth a surprising behaviour of the higher order moments of these kernels.

1. The kernel of *Rogosinski* (cf. e.g. [1]), a trigonometric polynomial of degree n, as given by

$$R_n(x)=\tfrac{1}{2}\left\{D_n\left(x+\frac{\pi}{2n+1}\right)+D_n\left(x-\frac{\pi}{2n+1}\right)\right\}$$

with $D_n(x)$ being Dirichlet's 'kernel' (indeed not a kernel in our sense)

$$D_n(x)=\tfrac{1}{2}\frac{\sin((2n+1)x/2)}{\sin(x/2)}$$

and convergence factors

$$\varrho_{k,n}(\mathfrak{N})=\begin{cases}\cos\dfrac{k\pi}{2n+1} & (1\leqslant k\leqslant n),\\ 0 & (k>n)\end{cases}$$

has $(2n-2)$ changes of sign in $(-\pi, \pi)$ and one double zero at $x=\pi$. Thus it is of infinite oscillation as $n \to \infty$. In view of

$$1-\varrho_{k,n}(\mathfrak{R}) = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{(2j)!} \left(\frac{k\pi}{2n+1}\right)^{2j} = \frac{\pi^2}{8} k^2 \frac{1}{n^2} + o(n^{-2}) \qquad (n \to \infty)$$

(hence (16) is obviously true) together with (7) and [13, p. 62 (36/37)]

$$(23) \qquad 2\sum_{k=1}^{\nu} (-1)^{k+1} \binom{2\nu}{\nu-k} k^{2i} = \begin{cases} \binom{2\nu}{\nu} & (i=0), \\ 0 & (i=1, 2, \ldots, \nu-1), \\ (2\nu)!(-1)^{\nu+1} & (i=\nu) \end{cases}$$

it follows that (cf. [14])

$$T(\mathfrak{R}; 2\sigma; n) = \frac{\pi^{2\sigma}}{(2n+1)^{2\sigma}} + o(n^{-2\sigma}) \qquad (n \to \infty).$$

This gives the positive limit

$$(24) \qquad \lim_{n\to\infty} n^{2\sigma} T(\mathfrak{R}; 2\sigma; n) = \left(\frac{\pi}{2}\right)^{2\sigma} \qquad (\sigma = 1, 2, 3, \ldots);$$

in this case, an increasing order 2σ of the moments effects that the corresponding order of approximation is monotonely increasing in the same way.

2. The *typical means of order* 2 ([1]) are defined by the convergence factors

$$\varrho_{k,n}(\mathfrak{M}) = 1 - \frac{k^2}{(n+1)^2} \qquad (1 \leqslant k \leqslant n).$$

Again by means of (23) one has for this polynomial kernel

$$T(\mathfrak{M}; 2\sigma; n) = \frac{2}{(n+1)^2} \sum_{k=1}^{\sigma} (-1)^{k+1} \binom{2\sigma}{\sigma-k} k^2 = \begin{cases} \dfrac{2}{(n+1)^2} & (\sigma=1), \\ 0 & (\sigma = 2, 3, 4, \ldots), \end{cases}$$

thus

$$(25) \qquad \begin{cases} \lim\limits_{n\to\infty} n^2 T(\mathfrak{M}; 2; n) = 2, \\ T(\mathfrak{M}; 2\sigma; n) = 0 & (\sigma = 2, 3, 4, \ldots). \end{cases}$$

Here all trigonometric moments of order 2σ, $\sigma \neq 1$, vanish. For an interpretation of this phenomenon from a saturation theoretical point of view compare [1, p. 475–478].

3. The final example is given by a certain *generalization of Abel-Poisson*'s kernel in [15], namely by

$$A_r(x) = p_r^2(x) \frac{1-r^2}{r} \cos x \qquad (r \to 1-)$$

with $p_r(x)$ being the original Abel-Poisson kernel

$$p_r(x)=\tfrac{1}{2}\,\frac{1-r^2}{1-2r\cos x+r^2}>0,\ \varrho_{k,r}(\mathfrak{P})=r^k$$

and convergence factors

$$\varrho_{k,r}(\mathfrak{A})=r^k\left\{1+k\,\frac{1-r^4}{4r^2}\right\}\qquad (k=1,2,3,\dots). \tag{26}$$

This is a kernel of finite oscillation with two fixed simple zeros at $x=\pm\pi/2$ and saturation limit

$$\lim_{r\to 1-}\frac{1-\varrho_{k,r}(\mathfrak{A})}{(1-r)^2}=\tfrac{1}{2}k^2.$$

Explicit formulae for the leading three moments are $(r\to 1-)$

$$T(\mathfrak{A};2;r)=\frac{r^2+2r+1}{2r}(1-r)^2,$$

$$T(\mathfrak{A};4;r)=-\frac{r+2}{r}(1-r)^4,$$

$$T(\mathfrak{A};6;r)=-\frac{3r^3-3r^2-7r+15}{2r}(1-r)^3.$$

More generally one has in the limit notation

$$\begin{cases}\displaystyle\lim_{r\to 1-}\frac{T(\mathfrak{A};2;r)}{(1-r)^2}=1\\[2ex] \displaystyle\lim_{r\to 1-}\frac{T(\mathfrak{A};4;r)}{(1-r)^4}=-3\\[2ex] \displaystyle\lim_{r\to 1-}\frac{T(\mathfrak{A};2\sigma;r)}{(1-r)^3}=-\tfrac{2}{3}\sum_{k=1}^{\sigma}(-1)^{k+1}\binom{2\sigma}{\sigma-k}k(k^2+2)=-4\binom{2\sigma-4}{\sigma-3}\\ \hfill(\sigma=3,4,5,\dots)\end{cases} \tag{27}$$

the latter relation follows from (7) and (26) using first de L'Hospital's rule in connection with (23) and, finally, for the right-hand equation (see [10, p. 39 (37, 38)])

$$\sum_{k=1}^{\sigma}(-1)^{k+1}\binom{2\sigma}{\sigma-k}k^j=\begin{cases}2\dbinom{2\sigma-3}{\sigma-2}=\dbinom{2\sigma-2}{\sigma-1} & (j=1;\ \sigma>1)\\[2ex] -\dfrac{2}{\sigma-2}\dbinom{2\sigma-4}{\sigma-3} & (j=3;\ \sigma\geqslant 3).\end{cases}$$

The few moment formulae (24), (25), and (27) reveal that, in comparison with positive kernels which generally satisfy the simple estimate (22), for instance, the situation for oscillating kernels is extremely different. This fact should give rise, apart from the Theorem, to a more refined characterization of the intrinsic common structure of this class of kernels. Here we only remark that, at least in our examples, the particular limit related to the second trigonometric moment and to the saturation order of the corresponding singular integral is always positive. Moreover, since most of the important positive kernels are 'k^2-kernels', this might be a first indication that the above kernels, thus the oscillating kernels of the Theorem, behave, in a certain sense, like positive kernels *).

Further consequences of the basic Lemma in connection with an improvement of the approximation order by means of finite oscillation kernels, as treated in [14], will appear in a forthcoming paper.

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*) There is another new paper of P. P. Korovkin [16] on Voronovskaja type expansions which, following the short review in Ref. Ž. Mat. 6B 102 (1970), might be connected with these subjects. Unfortunately the author was unable to obtain a copy of the original article.

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